# SOME CONTRIBUTIONS TO THE OPTIMUM ALLOCATION IN STRATUM SAMPLING

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## First part. Optimum Allocation in one-stage Stratum Sampling

I. *Signes used in this part.*

$L$ : Number of the strata in a population.

$N_i$ : Number of the units of analysis in the $i$-th stratum. $i=1, 2, 3, \cdots\cdots, L$.

$n_i$ : Number of the units of analysis drawn from the $i$-th stratum.

$k_i$ : Cost to inquuire one unit of analysis in the $i$-th stratum.

$T$ : Cost function.

$\sigma_i^2$ : Variance between the units of analysis in the $i$-th stratum.

$X'$ : Unbiased estimate of the total sum of the metrics of the units of analysis in a population.

$V(X')$: Variance of $X'$.

$\lambda$ : Lagrange multiplier.

II. *The case where $V(X')$ and $T$ are given by the following equations.*

$$V(X')=\sum_{i=1}^{L} N_i^2\frac{\sigma_i^2}{n_i}-\sum_{i=1}^{L} N_i\sigma_i^2. \tag{1}$$

$$T=\sum_{i=1}^{L} k_i\sqrt{n_i}\,. \tag{2}$$

II. A. *To find the values of $n_1, n_2, \cdots\cdots, n_L$ which minimize $V(X')$ keeping $T$ constant.*

Let

$$f(n_1,n_2,\cdots,n_L)=\sum_{i=1}^{L} N_i^2\frac{\sigma_i^2}{n_i}-\sum_{i=1}^{L} N_i\sigma_i^2+\lambda\left(\sum_{i=1}^{L} k_i\sqrt{n_i}-T\right). \tag{3}$$

Differentiating (3) by $n_1, n_2\cdots, n_L$ and then putting them equal to zero we get the simultaneous equations (4).

$$\lambda\frac{k_i}{2n_i^{\frac{1}{2}}}=\frac{N_i\sigma_i^2}{n_i^2}\ ,\quad i=1, 2, 3, \cdots, L. \tag{4}$$

From these equations we get

$$n_i=\frac{1}{\lambda^{\frac{2}{3}}}\left(\frac{2N_i^2\sigma_i^2}{k_i}\right)^{\frac{2}{3}},\quad i=1, 2, 3, \cdots, L. \tag{5}$$

Substituting (5) to (2) we get

$$T=\sum_{i=1}^{L} k_i\frac{1}{\lambda^{\frac{1}{3}}}\left(\frac{2N_i^2\sigma_i^2}{k_i}\right)^{\frac{1}{3}}=\frac{1}{\lambda^{\frac{1}{3}}}\sum_{i=1}^{L}\left(\frac{2N_i^2\sigma_i^2}{k_i}\right)^{\frac{1}{3}}k_i$$

$$\therefore \quad \frac{1}{\lambda^{\frac{2}{3}}} = \frac{T^2}{\left\{\sum_{i=1}^{L}\left(\frac{2N_i^2\sigma_i^2}{k_i}\right)^{\frac{1}{3}} k_i\right\}^2} = \frac{T^2}{\left\{\sum_{i=1}^{L}\left(\sqrt{2}\,k_iN_i\sigma_i\right)^{\frac{2}{3}}\right\}^2} \qquad (6).$$

Substituting (6) to (5) we get

$$n_i = \frac{T^2}{\left\{\sum_{i=1}^{L}\left(\sqrt{2}\,k_iN_i\sigma_i\right)^{\frac{2}{3}}\right\}^2}\left(\frac{2N_i^2\sigma_i^2}{k_i}\right)^{\frac{2}{3}}. \qquad (7).$$

Substituting (7) to (1) we get

$$V(X') = \sum_{i=1}^{L} N_i^2\sigma_i^2 \frac{\left\{\sum_{i=1}^{L}\left(k_iN_i\sigma_i\right)^{\frac{2}{3}}\right\}^2}{T^2}\left(\frac{k_i}{N_i^2\sigma_i^2}\right)^{\frac{2}{3}} - \sum_{i=1}^{L} N_i\sigma_i^2. \qquad (8).$$

Especially when $k_1 = k_2 = \cdots\cdots = k_L = k$ we get

$$n_i = \frac{T^2(2N_i^2\sigma_i^2)^{\frac{2}{3}}}{k^2\left\{\sum_{i=1}^{L}\left(\sqrt{2}\,N_i\sigma_i\right)^{\frac{2}{3}}\right\}^2}, \quad i=1,\ 2,\cdots\cdots,L. \qquad (9).$$

I. B. *To find the values of* $n_1, n_2, \cdots\cdots, n_L$ *which minimize* $T$ *keeping* $V(X')$ *constant.*

Let

$$f(n_1, n_2, \cdots\cdots, n_L) = \sum_{i=1}^{L} k_i\sqrt{n_i} + \lambda\left(\sum_{i=1}^{L} N_i^2\frac{\sigma_i^2}{n_i} - \sum_{i=1}^{L} N_i\sigma_i^2 - V(X')\right). \qquad (10).$$

Differentiating (10) by $n_1$, $n_2$, $\cdots\cdots$, $n_L$ and then putting them equal to zero, we get the simultaneous equations (11)

$$\frac{\partial f}{\partial n_i} = \frac{1}{2}k_in_i^{-\frac{1}{2}} + \lambda(-1)N_i^2\sigma_i^2n_i^{-2} = \frac{k_i}{2\sqrt{n_i}} - \lambda\frac{N_i^2\sigma_i^2}{n_i^2} = 0$$

$$\lambda\frac{N_i^2\sigma_i^2}{n_i^2} = \frac{k_i}{2\sqrt{n_i}}, \quad i=1,\ 2,\ 3,\cdots\cdots L. \qquad (11).$$

Erom these equations we get (12)

$$n_i = \lambda^{\frac{2}{3}}\left(\frac{2N_i^2\sigma_i^2}{k_i}\right)^{\frac{2}{3}}, \quad i=1,\ 2,\ 3,\cdots\cdots L. \qquad (12).$$

Substituting these relations to (1)

$$V(X') = \sum_{i=1}^{L} N_i^2\sigma_i^2\frac{1}{\lambda^{\frac{2}{3}}}\left(\frac{k_i}{2N_i^2\sigma_i^2}\right)^{\frac{2}{3}} - \sum_{i=1}^{L} N_i\sigma_i^2.$$

$$\lambda^{\frac{2}{3}} = \frac{\sum_{i=1}^{L}\left(\frac{k_iN_i^2\sigma_i^2}{4}\right)^{\frac{1}{3}}}{V(X') + \sum_{i=1}^{L} N_i\sigma_i^2} \qquad (13).$$

Substituting this relation to (12) we get

$$n_i = \frac{\sum_{i=1}^{L}\left(\frac{k_i N_i^2 \sigma_i^2}{4}\right)^{\frac{1}{3}}}{V(X') + \sum_{i=1}^{L} N_i \sigma_i^2}\left(\frac{2N_i^2\sigma_i^2}{k_i}\right)^{\frac{2}{3}}, \quad i=1, 2, \cdots\cdots, L. \tag{14}$$

Then $T$ is given by the following equation.

$$T = \sum_{i=1}^{L} k_i \sqrt{\frac{\sum_{i=1}^{L}\left(\frac{k_i N_i^2 \sigma_i^2}{4}\right)^{\frac{1}{3}}}{V(X') + \sum_{i=1}^{L} N_i \sigma_i^2}\left(\frac{2N_i^2\sigma_i^2}{k_i}\right)^{\frac{1}{3}}} \tag{15}$$

**Ⅲ.** *The case where $V(X')$ and $T$ are given by the following equations.*

$$V(X') = \sum_{i=1}^{L} N_i^2 \frac{\sigma_i^2}{n_i} - \sum_{i=1}^{L} N_i \sigma_i^2 \tag{1}$$

$$T = \sum_{i=1}^{L_1} k_i n_i + \sum_{j=L_1+1}^{L} k_j \sqrt{n_j}. \tag{2}$$

Ⅲ. A. *To find the values of $n_1, n_2, \cdots\cdots, n_L$ which minimize $V(X')$ keeping $T$ constant.*

Let

$$f(n_1, n_2, \cdots, n_L) = \sum_{i=1}^{L} N_i^2 \frac{\sigma_i^2}{n_i} - \sum_{i=1}^{L} N_i \sigma_i^2 + \lambda\left(\sum_{i=1}^{L_1} k_i n_i + \sum_{j=L_1+1}^{L} k_j \sqrt{n_j} - T\right) \tag{3}$$

Differentiating (3) by $n_1, n_2, \cdots\cdots, n_L$ we get (4).

$$\left.\begin{aligned} \frac{\partial f}{\partial n_i} &= N_i^2 \sigma_i^2 (-1) n_i^{-2} + \lambda k_i = -\frac{N_i^2 \sigma_i^2}{n_i^2} + \lambda k_i, \quad i = 1, 2, 3, \cdots\cdots, L_1. \\ \frac{\partial f}{\partial n_j} &= N_j^2 \sigma_j^2 (-1) n_j^{-2} + \lambda k_j \frac{1}{2} n_j^{-\frac{1}{2}} = -\frac{N_j^2 \sigma_j^2}{N_j^2} + \frac{\lambda k_j}{2 n_j^{\frac{1}{2}}}, \quad j = L_1 + 1, \cdots, L. \end{aligned}\right\} \tag{4}$$

The processes to solve the simultaneous equations

$$\left.\begin{aligned} -\frac{N_i^2 \sigma_i^2}{n_i^2} + \lambda k_i = 0, \quad i = 1, 2, \cdots\cdots, L_1, \\ -\frac{N_j^2 \sigma_j^2}{2 n_j^2} + \frac{\lambda k_j}{2 n_j^{\frac{1}{2}}} = 0, \quad j = L_1 + 1, \cdots\cdots L, \end{aligned}\right\} \tag{5}$$

are as follows.

From (5) we get

$$\left.\begin{aligned} n_i &= \sqrt{\frac{N_i^2 \sigma_i^2}{\lambda k_i}} = \frac{1}{\sqrt{\lambda}} \frac{N_i \sigma_i}{\sqrt{k_i}}, \quad i = 1, 2, \cdots\cdots, L_1. \\ n_j &= \frac{1}{\lambda^{\frac{2}{3}}}\left(\frac{2N_j^2 \sigma_j^2}{k_j}\right)^{\frac{2}{3}}, \quad j = L_1 + 1, \cdots\cdots, L. \end{aligned}\right\} \tag{6}$$

Substituting (6) to (1) we get (7),

$$T = \sum_{i=1}^{L_1} k_i \frac{1}{\sqrt{\lambda}} \frac{N_i \sigma_i}{\sqrt{k_i}} + \sum_{j=L_1+1}^{L} k_j \frac{1}{\lambda^{\frac{1}{3}}}\left(\frac{2N_j^2 \sigma_j^2}{k_j}\right)^{\frac{1}{3}}.$$

$$T = \frac{1}{\sqrt{\lambda}} \sum_{i=1}^{L_1} \sqrt{k_i} N_i \sigma_i + \frac{1}{\sqrt[3]{\lambda}} \sum_{j=L_1+1}^{L} (\sqrt{2} k_j N_j \sigma_j)^{\frac{2}{3}}. \tag{7}$$

Putting $\lambda^{\frac{1}{6}} = \rho$ we get an equation of $\rho$ that is (8).

$$\rho^3-\frac{B}{T}\rho-\frac{A}{T}=0. \tag{8}$$

where

$$A=\sum_{i=1}^{L_1}\sqrt{k_i}N_i\sigma_i, \quad B=\sum_{j=L_1+1}^{L}\left(\sqrt{2}k_jN_j\sigma_j\right)^{\frac{2}{3}}.$$

The method to get the first approximation is as follows.

$$\rho^3-\frac{B}{T}\rho-z=0.$$

$$3\rho^2d\rho-\frac{B}{T}d\rho-dz=0.$$

$$d\rho=\frac{1}{3\rho^2-\frac{B}{T}}dz.$$

Substituting $\sqrt{\frac{B}{T}}$ to $\rho$ and $\frac{A}{T}$ to $dz$ we get

$$\rho=\sqrt{\frac{B}{T}}+\frac{1}{3\frac{B}{T}-\frac{B}{T}}\frac{A}{T}=\frac{A}{2B}+\sqrt{\frac{B}{T}}. \tag{9}$$

This is the approximation of $\rho$.

Calulating the value of $\lambda$ by the relation $\lambda^{\frac{1}{6}}=\rho$ and then substituting it to (6) we can get the values of $n_1, n_2, \cdots\cdots, n_L$ which minimize the value of $T$ keeping $V(X')$ constant.

**Ⅲ. B.** *To find the values of $n_1, n_2\cdots\cdots, n_L$ which minimize $T$ keeping $V(X')$ constant.*

Let

$$f(n_1, n_2, \cdots\cdots, n_L)=\sum_{i=1}^{L_1}k_in_i+\sum_{j=L_1+1}^{L}k_j\sqrt{n_j}+\lambda\left(\sum_{i=1}^{L}N_i^2\frac{\sigma_i^2}{n_i}-\sum_{i=1}^{L}N_i\sigma_i^2-V(X')\right). \tag{10}$$

Differentiating (10) by $n_1, n_2, \cdots, n_L$ we get (11).

$$\left.\begin{aligned}\frac{\partial f}{\partial n_i}&=k_i+(-1)\lambda N_i^2\sigma_i^2\frac{1}{n_i^2}, \quad i=1,2,\cdots L_1.\\ \frac{\partial f}{\partial n_j}&=\frac{1}{2}k_jn_j^{-\frac{1}{2}}-\lambda N_j^2\frac{\sigma_j^2}{n_j^2}, \quad j=L_1+1,\cdots,L.\end{aligned}\right\} \tag{11}$$

Solving the simultaneous equations

$$\left.\begin{aligned}&k_i-\lambda N_i^2\sigma_i^2\frac{1}{n_i^2}=0, \quad i=1,2,\cdots L_1,\\ &\frac{k_j}{2n_j^{\frac{1}{2}}}-\lambda\frac{N_j^2\sigma_j^2}{n_j^2}=0, \quad j=L_1+1,\cdots L,\end{aligned}\right\} \tag{12}$$

we get (13).

$$\left.\begin{aligned}n_i&=\sqrt{\lambda}\sqrt{\frac{N_i^2\sigma_i^2}{k_i}}=\sqrt{\lambda}\frac{N_i\sigma_i}{\sqrt{k_i}}, \quad i=1,2,\cdots\cdots,L_1.\\ n_j&=\lambda^{\frac{2}{3}}\left(\frac{2N_j^2\sigma_j^2}{k_j}\right)^{\frac{2}{3}}, \quad j=L_1+1,\cdots\cdots,L.\end{aligned}\right\} \tag{13}$$

Substituting (13) to (2) we get (14).

$$V(X') = \sum_{i=1}^{L_1} N_i^2 \sigma_i^2 \frac{\sqrt{k_i}}{N_i \sigma_i} \frac{1}{\sqrt{\lambda}} + \sum_{j=L_1+1}^{L} N_j^2 \sigma_j^2 \left( \frac{k_j}{2N_j^2 \sigma_j^2} \right)^{\frac{2}{3}} \frac{1}{\lambda^{\frac{2}{3}}} - \sum_{i=1}^{L} N_i \sigma_i^2 .$$

$$V(X') = C\lambda^{-\frac{1}{2}} + D\lambda^{-\frac{2}{3}} - E. \qquad (14).$$

where

$$C = \sum_{i=1}^{L_1} \sqrt{k_i} N_i \sigma_i ,$$

$$D = \sum_{j=L_1+1}^{L} \left( \frac{k_j N_j \sigma_j}{2} \right)^{\frac{2}{3}} ,$$

$$E = \sum_{i=1}^{L} N_i \sigma_i^2 .$$

Putting $\lambda^{\frac{1}{6}} = \rho$ and $V(X') + E = v_0$ the equation (14) is transformed to (15).

$$\rho^4 - \frac{C}{v_0}\rho^3 - \frac{D}{v_0} = 0 . \qquad (15).$$

The method which was already explained is used to find the first approximation.

$$\rho^4 - \frac{C}{v_0}\rho^3 - z = 0 \qquad (16).$$

Differentiating (16) by $z$ we get (17).

$$d\rho = \frac{1}{\rho^2 \left( 4\rho - \frac{3C}{v_0} \right)} dz \qquad (17).$$

Substituting $\frac{C}{v_0}$ to $\rho$ and $\frac{D}{v_0}$ to $z$, the approximation is obtained by (18).

$$\rho = \frac{C}{v_0} + \frac{1}{\left( \frac{C}{v_0} \right)^2 \left( 4\frac{C}{v_0} - 3\frac{C}{v_0} \right)} \left( \frac{D}{v_0} \right) = \frac{C}{v_0} + \frac{v_0^2 D}{C^3} . \qquad (18).$$

Calculating $\lambda$ by the relation $\lambda = \rho^6$ and then substituting it to (13), $n_1, n_2, \cdots\cdots, n_L$ are obtained.

## Second part. Optimum allocation in two-stages stratum sampling

I. *Signs used in this part.*

$L$ : Number of the strata in a population.

$M_i$ : Number of the clusters (primary sampling units) contained in the $i$ -th stratum.

$m_i$ : Number of the colusters drawn from the $i$ -th stratum.

$N_{ij}$ : Number of the units of analysis (second sampling units) contained in the $j$ -th cluster of the $i$ -th stratum.

$n_{ij}$ : Number of the units of analysis drawn from the $j$ -th cluster in the $i$ -th stratum.

$f_i$ : Sampling ratio of the clusters in the $i$ -th stratum.

$g_{ij}$ : Sampling ratio of the units of analysis in the $j$ -th cluster of the $i$ -th stratum.

$\sigma_{ij}^2$ : $\sigma_{ij}^2 = \frac{1}{N_{ij}} \sum_{k=1}^{N_{ij}} (x_{ijk} - \bar{x}_{ij})^2 ,$

where $x_{ijk}$ is the metric of the $k$ -th unit of analysis in the $j$ -th cluster of the $i$ -th stratum, and $\bar{x}_{ij}=\frac{1}{N_{ij}}\sum_{k=1}^{N_{ij}} x_{ijk}$.

$\sigma_{ie}^2$ : $\sigma_{ie}^2=\frac{1}{M_i}\sum_{j=1}^{Mi}(X_{ij}-\bar{X}_i)^2$,

where

$$X_{ij}=\sum_{k=1}^{N_{ij}} x_{ijk},\quad \bar{X}_i=\frac{1}{M_i}\sum_{j=1}^{N_{ij}} X_{ij}$$

$k_i$ : Cost necessary to prepare the frame of the $i$ -th stratum.

$k_{ij}$ : Cost necessary to inquire one unit of analysis in the $j$ -th cluster of the $i$ -th stratum.

II. *The case where* $\frac{m_i}{M_i}=f$ *is independent of* $i$ *and* $\frac{n_{ij}}{N_{ij}}=g$ *is independent of* $i$ *and* $j$.

The well known formula

$$V(X')=\sum_{i=1}^{L}\left\{M_i^2\frac{M_i-m_i}{M_i-1}\frac{\sigma_{ie}^2}{m_i}+\frac{M_i}{m_i}\sum_{j=1}^{Mi}N_{ij}^2\frac{N_{ij}-n_{ij}}{N_{ij}-1}\frac{\sigma_{ij}^2}{n_{ij}}\right\}$$

and

$$T=\sum_{i=1}^{L}\left\{k_i m_i+\sum_{j=1}^{Mi}k_{ij}m_i n_{ij}\right\}$$

are transformed as follows.

$$V(X')=\sum_{i=1}^{L}\left\{M_i\frac{1}{f}\frac{1-\frac{1}{f}}{1-\frac{1}{M_i}}\sigma_{ie}^2+\frac{1}{f}\sum_{j=1}^{Mi}N_{ij}\frac{1}{g}\frac{1-g}{1-\frac{1}{N_{ij}}}\sigma_{ij}^2\right\}$$

$$=\sum_{i=1}^{L}\left\{\frac{1-f}{f}\frac{M_i^2}{M_i-1}\sigma_{ie}^2+\frac{1-g}{fg}\sum_{j=1}^{Mi}\frac{N_{ij}^2}{N_{ij}-1}\sigma_{ij}^2\right\}$$

$$=\frac{1-f}{f}A+\frac{1-g}{fg}B=(f^{-1}-1)A+f^{-1}(g^{-1}-1)B, \tag{1}$$

where

$$A=\sum_{i=1}^{L}\frac{M_i^2}{M_i-1}\sigma_{ie}^2,\quad B=\sum_{i=1}^{L}\sum_{j=1}^{Mi}\frac{N_{ij}^2}{N_{ij}-1}\sigma_{ij}^2,$$

and

$$T=f\sum_{i=1}^{L}k_iM_i+\sum_{i=1}^{L}\sum_{j=1}^{Mi}k_{ij}fgM_iN_{ij}=fC+fgD, \tag{2}$$

where

$$C=\sum_{i=1}^{L}k_iM_i,\quad D=\sum_{i=1}^{L}\sum_{j=1}^{Mi}k_{ij}M_iN_{ij}.$$

Ⅱ. A. *To find the values of $f$ and $g$ which minimize $V(X')$ keeping $T$ constant.*

Let

$$F(f,g)=(f^{-1}-1)A+f^{-1}(g^{-1}-1)B+\lambda(Cf+Dfg-T).$$

Then

$$\frac{\partial F}{\partial f}=\frac{-A}{f^2}-\frac{B}{f^2}\left(\frac{1}{g}-1\right)+\lambda(C+Dg). \qquad (3).$$

$$\frac{\partial F}{\partial g}=-\frac{1}{fg^2}B+\lambda Df. \qquad (4).$$

Putting $\frac{\partial F}{\partial g}=0$ we obtain $\lambda=\frac{B}{Df^2g^2}$. Substituting this relation to $\frac{\partial F}{\partial f}=0$ we get (5).

$$-\frac{A}{f^2}-\frac{1}{f^2}\left(\frac{1}{g}-1\right)B+\frac{B}{Df^2g^2}(C+Dg)=0.$$

$$(A-B)g^2=\frac{BC}{D}.$$

Assuming $A-B>0$, $g=\sqrt{\frac{BC}{(A-B)D}}$ (5).

Substituting (5) to (2) we get (6).

$$T=fC+f\sqrt{\frac{BC}{(A-B)D}}D.$$

$$f=\frac{T}{C+\sqrt{\frac{BCD}{A-B}}} \qquad (6).$$

Ⅱ. B. *To find the values of $f$ and $g$ which minimize $T$ keeping $V(X')$ constant.*

Let

$$F(f,g)=fC+fgD+\lambda\{(f^{-1}-1)A+f^{-1}(g^{-1}-1)B-V(X')\}.$$

Then

$$\frac{\partial F}{\partial f}=C+gD+\lambda\left(\frac{-A}{f^2}-\frac{1}{f^2}\,\frac{1-g}{g}\,B\right). \qquad (7).$$

$$\frac{\partial F}{\partial g}=fD-\lambda\frac{B}{fg^2}. \qquad (8).$$

Putting $\frac{\partial F}{\partial g}=0$ we get $\lambda=\frac{Df^2g^2}{B}$, and substituting this relation to $\frac{\partial F}{\partial f}=0$ we get (9).

$$C+gD+\frac{Df^2g^2}{B}\left(\frac{-A}{f^2}-\frac{1-g}{f^2g}B\right)=0.$$

Assuming $A-B>0$, $g=\sqrt{\frac{BC}{(A-B)D}}$. (9).

Substituting (9) to (1) we get

$$V(X')=\frac{1-f}{f}A+\frac{1}{f}\,\frac{1-\sqrt{\frac{BC}{(A-B)D}}}{\sqrt{\frac{BC}{(A-B)D}}}B.$$

Solving this equation by $f$ we get (10).

$$f=\frac{A+\left(\sqrt{\frac{(A-B)BD}{C}}-B\right)}{A+V(X')} \qquad (10).$$

**II.** *The special case where $\frac{m_i}{M_i}=f$ is independent of $i$ and $\frac{n_{ij}}{N_{ij}}=g_i$ is independent of $j$.*

In this case the formula about $V(X')$ and $T$ are transformed as follows.

$$V(X')=\frac{1-f}{f}\sum_{i=1}^{L}\left(\frac{M_i^2}{M_i-1}\right)\sigma_{ie}^2+\frac{1}{f}\sum_{i=1}^{L}\frac{1-g_i}{g_i}\sum_{j=1}^{Mi}\left(\frac{N_{ij}^2}{N_{ij}-1}\right)\sigma_{ij}^2$$

$$=(f_j^{-1}-1)A+f^{-1}\sum_{i=1}^{L}(g_i^{-1}-1)B_i\ , \qquad (1).$$

where $\quad B_i=\sum_{j=1}^{Mi}\left(\frac{N_{ij}^2}{N_{ij}-1}\right)\sigma_{ij}^2$ .

$$T=f\sum_{i=1}^{L}k_iN_i+\sum_{i=1}^{L}\sum_{j=1}^{Mi}k_{ij}fM_ig_iN_{ij}=fC+f\sum_{i=1}^{L}g_iD_i\ . \qquad (2).$$

**II. A.** *To find the values of $g_1, g_2, \cdots\cdots, g_L, f$ which minimize $V(X')$ keeping $T$ constant.*

Let

$$F(f,g_1,g_2,\cdots,g_L)=(f^{-1}-1)A+f^{-1}\sum_{i=1}^{L}(g_i^{-1}-1)B_i$$

$$+\lambda(fC+f\sum_{i=1}^{L}g_iD_i-T)\ . \qquad (3).$$

Then

$$\frac{\partial F}{\partial f}=\frac{-A}{f^2}-\frac{1}{f^2}\sum_{i=1}^{L}\left(\frac{1}{g_i}-1\right)B_i+\lambda C+\lambda\sum_{i=1}^{L}g_iD_i\ ,$$

$$\frac{\partial F}{\partial g_i}=-\frac{B_i}{fg_i^2}+\lambda fD_i\ , \quad i=1,2,\cdots\cdots L.$$

Putting $\frac{\partial F}{\partial g_i}=0$ we get (4) and (5).

$$\lambda=\frac{B_i}{f^2g_iD_i}, \qquad (4).$$

$$\frac{1}{f^2}=\frac{\lambda D_ig_i^2}{B_i}. \qquad (5).$$

From (4) it follows that

$$\frac{D_1g_1^2}{B_1}=\frac{D_2g_2^2}{B_2}=\cdots\cdots\cdots\cdots=\frac{D_Lg_L^2}{B_L}[=\rho^2]. \qquad (6).$$

Substituting (5) to $\frac{\partial F}{\partial f}=0$ we get the simultaneous equations (7) consist of $L$ equations.

$$-A\frac{\lambda D_i g_i^2}{B_i}-\frac{\lambda D_i g_i^2}{B_i}\sum_{i=1}^{L}\left(\frac{1}{g_i}-1\right)B_i+\lambda C+\lambda\sum_{i=1}^{L}g_i D_i=0$$

$$-\frac{AD_i}{B_i}g_i^2-\frac{D_i}{B_i}g_i^2\sum_{i=1}^{L}\left(\frac{1}{g_i}-1\right)B_i+C+\sum_{i=1}^{L}g_i D_i=0,\qquad i=1,2,\cdots,L. \tag{7}$$

From (6) we get (8).

$$g_i=\rho\sqrt{\frac{B_i}{D_i}}\,,\quad i=1,2,\cdots\cdots,L. \tag{8}$$

Substituting (8) to (7) we get a quadratic equation of $\rho$ (9).

$$-\frac{AD_i}{B_i}\rho^2\frac{B_i}{D_i}-\frac{D_i}{B_i}\rho^2\frac{B_i}{D_i}\sum_{i=1}^{L}\left(\frac{1}{\rho}\sqrt{\frac{D_i}{B_i}}-1\right)B_i+C+\sum_{i=1}^{L}\rho\sqrt{\frac{B_i}{D_i}}D_i=0.$$

$$\left(\sum_{i=1}^{L}B_i-A\right)\rho^2+\left\{\sum_{i=1}^{L}\left(\sqrt{B_iD_i}-\sqrt{\frac{D_i}{B_i}}\right)\right\}\rho+C=0. \tag{9}$$

Here assuming that $\left\{\sum_{i=1}^{L}\left(\sqrt{B_iD_i}-\sqrt{\frac{D_i}{B_i}}\right)\right\}^2-4C\left(\sum_{i=1}^{L}B_i-A\right)\geqq 0$

and $\sum_{i=1}^{L}B_i-A<0$

we get (10) as the positive root of (9).

$$\rho=\frac{\sum_{i=1}^{L}\left(\sqrt{\frac{D_i}{B_i}}-\sqrt{B_iD_i}\right)+\sqrt{\left\{\sum_{i=1}^{L}\left(\sqrt{\frac{D_i}{B_i}}-\sqrt{B_iD_i}\right)\right\}^2-4C\left(\sum_{i=1}^{L}B_i-A\right)}}{2\left(\sum_{i=1}^{L}B_i-A\right)}. \tag{10}$$

Substituting (10) to (8) $g_i$ is obtained to $i=1,2,\cdots\cdots L$.

From (2)

$$f=\frac{T}{C+\sum_{i=1}^{L}g_iD_i}. \tag{11}$$

Substituting the value of $g_i$, $i=1,2,\cdots\cdots,L$ to (11), $f$ is obtained.

**III. B.** *To find the values* of $g_1, g_2,\cdots,g_L$ and $f$ which minimize $T$ keeping $V(X')$ constant.

Let

$$F(f,g_1,\cdots,g_L)=fC+f\sum_{i=1}^{L}g_iD_i+\lambda\left\{(f^{-1}-1)A+f^{-1}\sum_{i=1}^{L}(g_i^{-1}-1)B_i\right\}.$$

Then

$$\frac{\partial F}{\partial f}=C+\sum_{i=1}^{L}g_iD_i+\lambda\left\{\frac{-A}{f^2}-\frac{1}{f^2}\sum_{i=1}^{L}(g_i^{-1}-1)B_i\right\},$$

$$\frac{\partial F}{\partial g_i}=fD_i-\lambda\frac{B_i}{fg_i^2},\quad i=1,2,\cdots\cdots L.$$

Putting $\frac{\partial F}{\partial g_i}=0$ we get (12) and (13).

$$\lambda=\frac{D_i f^2 g_i^2}{B_i}. \qquad (12).$$

$$\frac{1}{f^2}=\frac{D_i}{B_i\lambda}g_i^2. \qquad (13).$$

From (12) it follows that

$$\frac{D_1 g_1^2}{B_1}=\frac{D_2 g_2^2}{B_2}=\cdots\cdots\cdots=\frac{D_L g_L^2}{B_L}\left[\equiv\rho^2\right]. \qquad (14).$$

Substituting (13) to $\frac{\partial F}{\partial f}=0$ we get the simultaneous equations (15) consist of $L$ equations.

$$C+\sum_{i=1}^{L}g_iD_i+\left\{-\frac{AD_i}{B_i}g_i^2-\frac{D_i}{B_i}g_i^2\sum_{i=1}^{L}\left(\frac{1}{g_i}-1\right)B_i\right\}=0. \qquad (15).$$

$$[i=1,\ 2,\cdots\cdots L]$$

From (14) we get (16).

$$g_i=\rho\sqrt{\frac{B_i}{D_i}},\quad i=1,\ 2,\cdots\cdots L. \qquad (16).$$

Substituting (16) to (15) we get a quadratic equation of $\rho$ (17).

$$C+\sum_{i=1}^{L}\rho\sqrt{\frac{B_i}{D_i}}D_i+\left\{-\frac{AD_i}{B_i}\rho^2\frac{B_i}{D_i}-\frac{D_i}{B_i}\rho^2\frac{B_i}{D_i}\sum_{i=1}^{L}\left(\frac{1}{\rho}\sqrt{\frac{D_i}{B_i}}-1\right)B_i\right\}=0$$

$$\left(A-\sum_{i=1}^{L}B_i\right)\rho^2=C. \qquad (18).$$

Assuming $A-\sum_{i=1}^{L}B_i>0$ we get (18) as the positive root of (17).

$$\rho=\sqrt{\frac{C}{A-\sum_{i=1}^{L}B_i}} \qquad (19).$$

Substituting (18) to (16), $g_1, g_2, \cdots\cdots, g_L$ are obtained.

From (1) and (19) we get (20) which gives the value of $f$.

$$fV(X')=(1-f)A+\sum_{i=1}^{L}B_i\frac{1-g_i}{g_i}.$$

$$f\{V(X')+A\}=A+\sum_{i=1}^{L}\frac{B_i(1-g_i)}{g_i}.$$

$$f=\frac{B+\sum_{i=1}^{L}B_i\left(\frac{1}{g_i}-1\right)}{V(X')+A}.$$

$$= \frac{\left(A - \sum_{i=1}^{L} B_i\right) + \sum_{i=1}^{L}\left(\sqrt{\frac{A - \sum_{i=1}^{L} B_i}{C}} \sqrt{B_i D_i} - B_i\right)}{V(X') + A} \qquad (20).$$

Ⅳ. *The special case where $f_i$ and $g_{ij}$ are independent of $i$ and $j$, and $T$ is given by the following equation.*

$$T = \sum_{i=1}^{L}\left(k_i \sqrt{m_i} + \sum_{j=1}^{M_i} k_{ij} m_i n_{ij}\right)$$

$$= \sum_{i=1}^{L}\left(k_i \sqrt{fM_i} + \sum_{j=1}^{M_i} k_{ij} fM_i g N_{ij}\right)$$

$$= f^{\frac{1}{2}} \sum_{i=1}^{L} (k_i \sqrt{M_i}) + fg \sum_{i=1}^{L} \sum_{j=1}^{M_i} k_{ij} M_i N_i = f^{\frac{1}{2}} C + fgD,$$

$$C = \sum_{i=1}^{L} (k_i \sqrt{M_i}), \quad D = \sum_{i=1}^{L} \sum_{j=1}^{M_i} k_{ij} M_i N_{ij}. \qquad (2).$$

In this case, the equation which gives $V(X')$ is the same as (1) in I.

Ⅳ. A. *To find the values of $f$ and $g$ which minimize $V(X')$ keeping $T$ constant.*

Let

$$F(f, g) = (f^{-1} - 1)A + f^{-1}(g^{-1} - 1)B + \lambda(f^{\frac{1}{2}} C + fgD). \qquad (3).$$

Then $\dfrac{\partial F}{\partial f} = -\dfrac{A}{f^2} - \dfrac{B}{f^2}\left(\dfrac{1}{g} - 1\right) + \lambda\left(\dfrac{C}{2f^{\frac{1}{2}}} + gD\right).$

$$\frac{\partial F}{\partial g} = -\frac{B}{fg^2} + \lambda fD.$$

From $\dfrac{\partial F}{\partial g} = 0$ we get $\lambda = \dfrac{B}{Df^2g^2}$, and substituting this relation to $\dfrac{\partial F}{\partial f} = 0$ we get (4).

$$-\frac{A}{f^2} - \frac{1}{f^2}\left(\frac{1}{g} - 1\right)B + \frac{B}{Df^2g^2}\left(\frac{C}{2f^{\frac{1}{2}}} + gD\right) = 0.$$

$$\frac{B - A}{f^2} + \frac{BC}{2Df^{\frac{5}{2}} g^2} = 0.$$

$$(B - A)f^{\frac{1}{2}} + \frac{BC}{2Dg^2} = 0.$$

Assuming $A - B > 0$,

$$f^{\frac{1}{2}} = \frac{BC}{2D(A - B)g^2} \qquad (4).$$

Substituting (4) to (2) we get an equation of three degrees by $g$ (5).

$$(f^{\frac{1}{2}})^2 gD + f^{\frac{1}{2}} C - T = 0.$$

$$\frac{B^2C^2}{4D(A - B)^2 g^3} + \frac{BC^2}{2D(A - B)g^2} - T = 0.$$

$$Tg^3 - \frac{BC^2}{2D(A - B)} g - \frac{B^2C^2}{4D(A - B)^2} = 0.$$

$$g^3-\frac{BC^2}{2DT(A-B)}g-\frac{B^2C^2}{4DT(A-B)^2}=0. \tag{5}$$

The familiar method is used without explanation to find the first approximation.

$$g^3-\frac{BC^2}{2DT(A-B)}g-z=0.$$

$$dg=\frac{1}{3g^2-\dfrac{BC^2}{2DT(A-B)}}dz. \tag{6}$$

$$g=\sqrt{\frac{BC^2}{2DT(A-B)}}+\frac{1}{3\dfrac{BC^2}{2DT(A-B)}-\dfrac{BC^2}{2DT(A-B)}}\left\{\frac{B^2C^2}{4DT(A-B)^2}\right\}$$

$$=\sqrt{\frac{BC^2}{2DT(A-B)}}+\frac{B}{4(A-B)}. \tag{7}$$

$$f^{\frac{1}{2}}=\frac{-C+\sqrt{C^2+4gDT}}{2gD}$$

$$\therefore\ f=\frac{C^2-2C\sqrt{C^2+4gDT}+C^2+4gDT}{4g^2D^2}=\frac{2C^2+4gDT-2C\sqrt{C^2+4gDT}}{4g^2D^2}$$

$$=\frac{C^2+2gDT-C\sqrt{C^2+4gDT}}{2g^2D^2}. \tag{8}$$

Substituting (7) to (8) $f$ is obtained.

IV. B. *To find the values of $f$ and $g$ which minimize $T$ keeping $V(X')$ constant.*

Let

$$F(f,g)=f^{\frac{1}{2}}C+fgD+\lambda\{(f^{-1}-1)A+f^{-1}(g^{-1}-1)B-V(X')\}.$$

Then

$$\frac{\partial F}{\partial f}=\frac{C}{2f^{\frac{1}{2}}}+gD+\lambda\left\{-\frac{A}{f^2}-\frac{B}{f^2}\left(\frac{1}{g}-1\right)\right\},$$

$$\frac{\partial F}{\partial g}=fD-\lambda\frac{B}{fg^2}.$$

From $\dfrac{\partial F}{\partial g}=0$ we get $\lambda=\dfrac{Df^2g^2}{B}$, and substituting this relation to $\dfrac{\partial F}{\partial f}=0$ we get

$$\frac{1}{2}\frac{C}{f^{\frac{1}{2}}}+gD+\frac{D}{B}f^2g^2\left\{-\frac{A}{f^2}-\frac{B}{f^2}\left(\frac{1}{g}-1\right)\right\}=0.$$

$$\frac{C}{2f^{\frac{1}{2}}}-\frac{AD}{B}g^2+Dg^2=0.$$

$$f^{\frac{1}{2}}=\frac{BC}{2g^2D(A-B)}.$$

$$f=\frac{B^2C^2}{4D^2(A-B)^2g^4}. \tag{9}$$

Substituting (9) to (1) we get an equation of four degrees by $g$ (10).

$$V(X')=\left\{\frac{4D^2(A-B)}{B^2C^2}g^4-1\right\}A+\frac{4D^2(A-B)^2g^4}{B^2C^2}\left(\frac{1}{g}-1\right)B.$$

$$\frac{4D^2(A-B)^3}{B^2C^2}g^4+\frac{4BD^2(A-B)^2}{B^2C^2}g^3-\{A+V(X')\}=0. \tag{10}$$

Considering that $g$ is smaller than 1 and the ratio of the coefficient of $g^4$ to that of $g^3$, that is, $\dfrac{A}{B}-1$ is not so great, we can neglect the first term of (10). Solving this equation under this condition we get (11).

$$\frac{4BD^2(A-B)^2}{B^2C^2}g^3=A+V(X').$$

$$g^3=\frac{B^2C^2\{A+V(X')\}}{4BD^2(A-B)^2}.$$

$$\therefore\ g=\sqrt[3]{\frac{BC^2\{A+V(X')\}}{4D^2(A-B)^2}}. \qquad (11).$$

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